Problem Set 6
Optical Waveguides and Fibers (OWF)
will be discussed in the tutorial on December 09, 2015

Exercise 1: Modes at a dielectric-metal interface: Surface plasmon polaritons.

The real part of the dielectric constant can sometimes be negative, like in the case of metals at frequencies smaller than the plasma frequency. In this case it is possible to have guided modes that are confined to the boundary between the metal and a dielectric. The aim of this exercise is to describe these modes (called "surface plasmon polaritons") starting from the slab waveguide description by letting the thickness of the core tend to zero, see Fig. 1. Consider a slab waveguide defined by three layers having relative

Figure 1: Metal-dielectric interface.

dielectric constants $\underline{\epsilon}_3$ (top cladding), $\underline{\epsilon}_1$ (core) and $\underline{\epsilon}_2$ (bottom cladding). The layers are oriented parallel to the $yz-$plane and the thickness of the core layer is $2a$, as drawn in Fig. 1. For the guided fundamental TM-mode of an asymmetric slab waveguide, the propagation constant β is determined by the following equation:

$$u = \frac{1}{2}\arctan\left(\frac{\underline{\epsilon}_1 w}{\underline{\epsilon}_2 u}\right) + \frac{1}{2}\arctan\left(\frac{\underline{\epsilon}_1 w'}{\underline{\epsilon}_3 u}\right), \tag{1}$$

with the normalized parameters defined as

$$\begin{aligned} u &= a \cdot k_{1x} = a\sqrt{\underline{\epsilon}_1 k_0^2 - \beta^2} \\ w &= a \cdot k_2^{(i)} = a\sqrt{\beta^2 - \underline{\epsilon}_2 k_0^2} \\ w' &= a \cdot k_3^{(i)} = a\sqrt{\beta^2 - \underline{\epsilon}_3 k_0^2}. \end{aligned}$$

a) Let the thickness a tend to zero. Show that the following condition must be satisfied:

$$\frac{w}{\underline{\epsilon}_2} = -\frac{w'}{\underline{\epsilon}_3} \tag{2}$$

Solution:

Eq. (1) can be reformulated as

$$\frac{1}{2}\arctan\left(\frac{\underline{\epsilon}_1 w}{\underline{\epsilon}_2 u}\right) = -\frac{1}{2}\arctan\left(\frac{\underline{\epsilon}_1 w'}{\underline{\epsilon}_3 u}\right) - u,$$

with $a \to 0$ it follows $u = a \cdot k_{1x} \to 0$ and by using the point symmetry of the arctan we can write

$$\begin{aligned} -\frac{\underline{\epsilon}_1 w}{\underline{\epsilon}_2 u} &= \frac{\underline{\epsilon}_1 w'}{\underline{\epsilon}_3 u} \\ \frac{w}{\underline{\epsilon}_2} &= -\frac{w'}{\underline{\epsilon}_3} \\ \frac{\sqrt{\beta^2 - \underline{\epsilon}_2 k_0^2}}{\underline{\epsilon}_2} &= -\frac{\sqrt{\beta^2 - \underline{\epsilon}_3 k_0^2}}{\underline{\epsilon}_3} \end{aligned}$$

b) For a guided mode, w and w' must be real and positive. Assume further that both $\underline{\epsilon}_2$ and $\underline{\epsilon}_3$ are real. From Eq. (2) we find that $\underline{\epsilon}_2$ and $\underline{\epsilon}_3$ must have opposite sign in order to obtain a guided mode. Calculate the propagation constant β from Eq. (2). Show that a propagating mode with real β exists only if $\underline{\epsilon}_2 < -\underline{\epsilon}_3$.

Solution:

With Eq. (2) we can write the following equation:

$$\frac{w^2}{w'^2} = \frac{\beta^2 - \underline{\epsilon}_2 k_0^2}{\beta^2 - \underline{\epsilon}_3 k_0^2} = \frac{\underline{\epsilon}_2^2}{\underline{\epsilon}_3^2}$$
$$\underline{\epsilon}_3^2 \beta^2 - k_0^2 \underline{\epsilon}_2 \underline{\epsilon}_3^2 = \underline{\epsilon}_2^2 \beta^2 - k_0^2 \underline{\epsilon}_3 \underline{\epsilon}_2^2$$
$$\beta^2 \left(\underline{\epsilon}_3^2 - \underline{\epsilon}_2^2\right) = k_0^2 \left(\underline{\epsilon}_3^2 - \underline{\epsilon}_2^2\right) \frac{\underline{\epsilon}_2 \underline{\epsilon}_3}{\underline{\epsilon}_2 + \underline{\epsilon}_3}$$
$$\beta = k_0 \sqrt{\frac{\underline{\epsilon}_2 \underline{\epsilon}_3}{\underline{\epsilon}_2 + \underline{\epsilon}_3}}$$

Since the product $\underline{\epsilon}_2 \underline{\epsilon}_3$ in the square root is negative, the condition $\underline{\epsilon}_2 < -\underline{\epsilon}_3$ has to be fulfilled in order to get a negative denominator and thereby a real propagation constant β.

c) Identify all non-zero field components of the slab waveguide TM-mode along with the boundary conditions that these fields must fulfill at the metal-dielectric interface. All components must decay evanescently into the top and the bottom cladding. Sketch qualitatively all non-vanishing field components for $\underline{\epsilon}_3 = 1$, $\underline{\epsilon}_2 = -4$.

Solution:

The mode of interest is the fundamental TM mode. Therefore the non-zero field components $\underline{\mathcal{H}}_y(x)$, $\underline{\mathcal{E}}_x(x)$ and $\underline{\mathcal{E}}_z(x)$ have to fulfill Maxwell's equations. The boundary conditions are the continuity of $\underline{\mathcal{H}}_y(x)$, $\underline{\mathcal{E}}_z(x)$ and $\underline{\mathcal{D}}_x(x)$ at the interface. For the $\underline{\mathcal{H}}_y(x)$ field we find

$$\underline{\mathcal{H}}_y(x) = \begin{cases} H_0 e^{-k_3^{(i)} x} & \text{for} \quad x > 0 \\ H_0 e^{k_2^{(i)} x} & \text{for} \quad x < 0 \end{cases},$$

with $k_2^{(i)}$ and $k_3^{(i)}$ as the negative and positive imaginary parts of the transverse wave vector components in the metal and the dielectric region. The fields $\underline{\mathcal{E}}_x(x)$ and $\underline{\mathcal{E}}_z(x)$ are

$$\underline{\mathcal{E}}_x(x) = \frac{\beta}{\omega \epsilon_0} \begin{cases} \frac{1}{\underline{\epsilon}_3} \underline{\mathcal{H}}_y(x) & \text{for} \quad x > 0 \\ \frac{1}{\underline{\epsilon}_2} \underline{\mathcal{H}}_y(x) & \text{for} \quad x < 0 \end{cases},$$
$$\underline{\mathcal{E}}_z(x) = \mathrm{j} \frac{1}{\omega \epsilon_0} \begin{cases} \frac{k_3^{(i)}}{\underline{\epsilon}_3} \underline{\mathcal{H}}_y(x) & \text{for} \quad x > 0 \\ -\frac{k_2^{(i)}}{\underline{\epsilon}_2} \underline{\mathcal{H}}_y(x) & \text{for} \quad x < 0 \end{cases}.$$

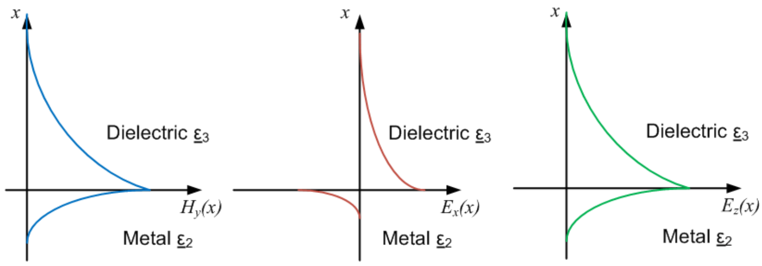

Figure 2: Sketches of the fields

Questions and Comments:

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